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Energies of some non-regular graphs

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The energy of a graph G is the sum of the absolute values of its eigenvalues. In this paper, we study the energies of some classes of non-regular graphs. Also the spectrum of some non-regular graphs and their complements are discussed.

KEY WORDS: eigenvalues, energy, equienergetic graphs

1. Introduction

Let G be a graph on p vertices with adjacency matrix A. Then A is a real symmetric matrix and so the eigenvalues of A are real and hence can be ordered. The eigenvalues of $A, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ are called the eigenvalues of G and form the spectrum of G. The energy $E(G)$ of a graph G is then defined as the sum of absolute values of its eigenvalues. That is $E(G) = \sum_{i=1}^{n} |\lambda_i|$. The study of properties of E was initiated by Gutman [5]. In chemistry, the energy of a graph is well studied [3], since it can be used to approximate the total π-electron energy of a molecule. In chemical graph theory an important line of research has been the search for approximate expressions or bounds for the total π-electron energy. There are a lot of results on the bound for E which pertain to special class of graphs most of which are regular [7].

In [4] the eigenvalue distribution of regular graphs, the spectra of some well known family of graphs, their energies and the relation between eigenvalues of a regular graph and its complement are studied. In [10], the energy of iterated line graphs of regular graphs are obtained and a family of regular equienergetic graphs are presented. In [2,12] the existence of a pair of equienergetic graphs on p vertices is proved for every $p \equiv 0 \bmod(4)$ and $p \equiv 0 \bmod(5)$ and in [9] we have extended the same for $p = 6, 14, 18$, and $p \geq 20$ and some other recent works are [6, 7, 10]. Some aspects of chemical applications of graph theory is discussed in [8].

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In this paper, the emphasis is on the energy of non-regular graphs. In the first part, we discuss energies of some classes of graphs arising from graph cross products. Using this we obtain some non-regular equienergetic graphs.

In the second part, we study some operations on a given graph G and the energy of the resultant graph in terms of the energy of G is obtained. Using these operations on regular graphs whose energy is known, we obtain energies of some non-regular family of graphs.

In the third part, we obtain the eigenvalues of complements of some non-regular graphs. All graph theoretic terminologies are from Ref. [1]. We use the following lemmas in this paper.

Lemma 1 [4]. Let M, N, P, and Q be matrices with M invertible. Let $S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$. Then $\det S = |M| \left|Q - PM^{-1}N\right|$.

Lemma 2 [4]. Let M, N, P, and Q be matrices. Let $S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$. If M and P commutes then $\det S = |MQ - PN|$.

Lemma 3 [4]. Let G be graph with spec(G)$=\{\lambda_i\}$, $i = 1$ to n and H a graph with spec(H)$=\left\{\mu_j\right\}$, $j = 1$ to n'. Then the spectrum of cartesian product of G and H is given by $\text{spec}(G \times H) = \left\{\lambda_i + \mu_j\right\}$, $i = 1$ to n, $j = 1$ to n'.

Lemma 4 [10]. Let G be an r regular graph with $spec(G)=\{\lambda_i\}$, $i = 1$ to p.Then the spectrum of $L^2(G)$ is given by

$$\begin{pmatrix} 4r-6 & \lambda_2+3r-6 & \dots & \lambda_p+3r-6 & 2r-6 & -2 \\ 1 & 1 & \dots & 1 & \frac{p(r-2)}{2} & \frac{pr(r-2)}{2} \end{pmatrix}.$$

Lemma 5 [4]. The spectrum of K_m is $\begin{pmatrix} m-1 & -1 \\ 1 & m-1 \end{pmatrix}$.

Lemma 6 [4]. Let G be an $r-$ regular graph on p vertices with $r = \lambda_1, \lambda_2, \dots, \lambda_m$ as the distinct eiegenvalues. Then there exists a polynomial $P(x)$ such that $P\{A(G)\} = J$ where J is the all one matrix of order p and $P(x)$ is given by

$$P(x) = p \times \frac{(x-\lambda_2)(x-\lambda_3)\dots(x-\lambda_m)}{(r-\lambda_2)(r-\lambda_3)\dots(r-\lambda_m)},$$

so that $P(r) = p$ and $P(\lambda_i) = 0$ for all $\lambda_i \neq r$.

Lemma 7 [4]. Let A be a matrix with λ as an eiegenvalue. Then for any polynomial $f(x)$, $f(\lambda)$ is an eigenvalue of $f(A)$.

2. Energy of Cartesian product of some graphs

In this section, we first consider some graphs whose spectrum is contained in $[-2k, 2k]$ for some k and then use it to construct non-regular equienergetic graphs.

Example

1. For any $2k$ regular graph G, the spectrum of all vertex deleted subgraphs $G - v$ lies in $[-2k, 2k]$.
2. G and H are two graphs on five vertices whose spectrum is contained in $[-4, 4]$. See figure 1.

Notation:

Let G be a graph. Then G^k denote the cross product of G, k times.

Theorem 1. Let G be an r regular graph on p vertices with $r \geq 2(k+1)$. Then for any graph F on n vertices whose spectrum is contained in $[-2k, 2k]$,

$$E\left[\left\{L^2(G)\right\}^k \times F\right] = \frac{nk}{2^{k-2}}\left[pr(r-2)\right]^k .$$

Proof. By lemmas 3 and 4 the only negative eigenvalues of $\left\{L^2(G)\right\}^k$ is $-2k$ with multiplicity $\left[\frac{pr(r-2)}{2}\right]^k$ for $r \geqslant k+2$.

Let F be a graph with spectrum contained in $[-2k, 2k]$. Then by lemma 3, for $r \geq 2(k+1)$, the only negative eigenvalues of $\left[\left\{L^2(G)\right\}^k \times F\right]$ are $-2k + \mu_i$, where $\mu_i, i = 1$ to n are the eigenvalues of F, each with multiplicity $\left[\frac{pr(r-2)}{2}\right]^k$. Thus by definition of energy, we get

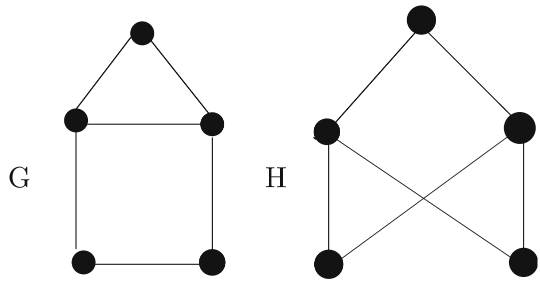

Figure 1. Two graphs whose spectrum is contained in [−4, 4].

$$E\left[\left\{L^2(G)\right\}^k \times F\right] = 2 \times \left[\frac{pr(r-2)}{2}\right]^k \sum_{i=1}^{n} |-2k + \mu_i|$$
$$= \frac{nk}{2^{k-2}}\left[pr(r-2)\right]^k . \qquad \square$$

Corollary 1. For any $r \geqslant 4$ regular p point graph G, $L^2(G) \times C_n$ and $L^2(G) \times P_n$ are equienergetic with energy $2pnr(r-2)$.

Proof. Proof follows from the fact that the spectra of C_n and P_n lies in $[-2, 2]$. $\square$

Corollary 2. For any $r \geqslant 4$ regular graph G, $L^k(G) \times C_n$ and $L^k(G) \times P_n$ are equienergetic for $k \geqslant 3$.

Proof. Since $L^q(G) = L^2[L^{q-2}(G)]$, the claim follows from corollary 1. $\square$

Corollary 3. Let F_1 and F_2 be non-isomorphic, non-regular graphs on n vertices whose spectrum is contained in $[-2k, 2k]$. Then $L^k(G) \times F_1$ and $L^k(G) \times F_2$ are non-regular and equienergetic with energy $\frac{nk}{2^{k-2}}\left[pr(r-2)\right]^k$.

Theorem 2. Let m and k be positive integers with $m \geqslant 2k$. Then for any graph G on p vertices whose spectrum is contained in $[-k, k]$,$E\left[\{K_m\}^k \times G\right] = 2pk(m-1)^k$.

Proof. From lemma 3 it follows that the spectrum of $\{K_m\}^k$ is

$$\begin{pmatrix} km-k & (k-1)m-k & (k-2)m-k & \dots & m-k & -k \\ 1 & kC_1(m-1) & kC_2(m-1)^2 & \dots & kC_1(m-1)^k & (m-1)^k \end{pmatrix}.$$

Now, given that G is a graph on p vertices whose spectrum is contained in $[-k, k]$. Thus for every $\mu_i \in$spec(G), we have $\mu_i + k \geq 0$. Thus if $m \geq 2k$ then by lemma 3 the only negative eigenvalues of $\{K_m\}^k \times G$ is $-k + \mu_i$,i =1 to p each with multiplicity $(m-1)^k$. Thus

$$E\left[\{K_m\}^k \times G\right] = 2 \times (m-1)^k \times \sum_{i=1}^{p} |-k + \mu_i|$$
$$= 2pk(m-1)^k. \qquad \square$$

Corollary 4. $(K_m \times K_m) \times C_n$ and $(K_m \times K_m) \times P_n$ are equienergetic with energy $4n(m-1)^2$.

Corollary 5. Let F_1 and F_2 be non-isomorphic, non-regular graphs on p vertices whose spectrum is contained in $[-k, k]$. Then for every $m \geq 2k$,$\{K_m\}^k \times F_1$ and $\{K_m\}^k \times F_2$ are non-regular equienergetic graphs on $m^k p$ vertices with energy $2pk(m-1)^k$.

3. Energy of some classes of non-regular graphs

Definition 1 [11]. Let G be a graph on p vertices labelled as $V = \{v_1, v_2, v_3, \ldots, v_p\}$. Then take another set $U = \{u_1, u_2, \ldots, u_p\}$ of p vertices. Now define a graph H with $V(H) = V \bigcup U$ and edge set of H consisting only of those edges joining u_i to neighbors of v_i in G for each i. The resultant graph H is called the identity duplication graph of G denoted by DG.

Let G be a connected r-regular graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$. We shall now consider the following seven operations on G, denote the resultant non-regular graphs by H_i,$i = 1, 2 \ldots 7$ and obtain expressions for the energies of these graphs in terms of the energy of G.

Operation 1. Let G_1 be the identity duplication graph of G. Then introduce k new vertices and join each of these k new vertices to all vertices of G only.

Operation 2. Introduce two sets $U = \{u_i\}$ and $W = \{w_i\}$ of p vertices and make u_i adjacent to vertices in $N(v_i)$ and w_i adjacent to vertices in $\overline{N(v_i)}$.

Operation 3. Introduce one copy of G on $U = \{u_i\}$. Make u_i adjacent to those vertices in $\overline{N(v_i)}$ for each i.

Operation 4. Introduce two sets $U = \{u_i\}$,$i = 1, 2, \ldots, p$ and $W = \{w_j\}$, $j = 1, 2, \ldots, k$. Now make u_i adjacent to all vertices in $\overline{N(v_i)}$ for each i and join every vertex of W to all vertices of G.

Operation 5. Introduce two sets $U = \{u_i\}$ and $W = \{w_i\}$ of p vertices each and make u_i adjacent to vertices in $\overline{N(v_i)}$ and w_i adjacent to vertices in $\overline{N(v_i)}$.

Operation 6. Introduce two sets $U = \{u_i\}$ and $W = \{w_i\}$ of p vertices each. Then join u_i to vertices in $N(v_i)$ and w_i to vertices in $\overline{N(v_i)}$ or each i and remove the edges of G.

Operation 7. Introduce a set $U = \{u_i\}$ of p vertices. Then join u_i to vertices in $\overline{N(v_i)}$ for each i.Then take a set W of k vertices and join each of them to all vertices of G and remove the edges of G.

Theorem 3. Let G be a connected r regular graph and $H_i, i = 1, 2, \ldots, 7$ be the graphs described as above. Then

$$E(H_1) = 2\left[E(G) - r + \sqrt{r^2 + pk}\right],$$

$$E(H_2) = 3\,(E(G) - r) + \sqrt{r^2 + 4\left\{(p-r)^2 + r^2\right\}},$$

$$E(H_3) = \begin{cases} 2\,[E(G) + p - 2r], & \text{if } p \geq 2r, \\ 2E(G), & \text{if } p < 2r, \end{cases}$$

$$E(H_4) = \sqrt{5}\,[E(G) - r] + \sqrt{r^2 + 4\left(pk + \{p-r\}^2\right)},$$

$$E(H_5) = 3\,[E(G) - r] + \sqrt{r^2 + 8\,(p-r)^2},$$

$$E(H_6) = 2\left\{\sqrt{2}\,(E(G) - r) + \sqrt{r^2 + (p-r)^2}\right\},$$

$$E(H_7) = 2\left[E(G) - r + \sqrt{(p-r)^2 + pk}\right].$$

Proof. In each of the operations, using lemmas 1, 6, and 7, the characteristic polynomial and the eigenvalues are given in table 1.

Now the expressions for the energies follows from column 4 of table 1. □

4. Eigenvalues of complements of some non-regular graphs

Let G be an $r-$regular graph on p vertices with spectrum $\{\lambda_i\}_{i=1}^{p}$. Then by Cvetkovic et al. [4] the eigenvalues of $\overline{G}$ are $p - r - 1$ and $-1 - \lambda_i$ where λ_i is an eigenvalue of G different from r. However, no such relation exists between the eigenvalues of a non-regular graph and its complement.

In this section, we give the eigenvalues of some non-regular graphs and their complements obtained using the following operations on regular graphs.

Let G be a connected $r-$regular graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$. Consider the following operations on G and denote the resultant graphs by $F_i, i = 1, \ldots, 8$.

Operation 8. Introduce a copy of $\overline{G}$ on $U = \{u_1, u_2, \ldots, u_p\}$. Make u_i adjacent to v_i.

Operation 9. Introduce a copy of G on $U = \{u_1, u_2, \ldots, u_p\}$. Make u_i adjacent to vertices in $\overline{N[v_i]}$.

Table 1
Spectrum of $H_i s$.

Op:	Adjacency matrix	Ch: Polynomial	Eigenvalues
1	$\begin{bmatrix} 0_p & A & J_{p\times k} \\ A & 0_p & 0_{p\times k} \\ J_{k\times p} & 0_{k\times p} & 0_k \end{bmatrix}$	$x^k \prod_{i=1}^{p} \left[x^2 - kP(\lambda_i) - \lambda_i^2\right]$	$x = 0$;k times $= \pm\sqrt{r^2 + pk}$ $= \pm\lambda_i; \lambda_i \neq r$
2	$\begin{bmatrix} A & A & \overline{A}+I \\ A & 0_p & 0_p \\ \overline{A}+I & 0_p & 0_p \end{bmatrix}$	$x^p \prod_{i=1}^{p} \left[x(x-\lambda_i) - (P(\lambda_i) - \lambda_i)^2 - \lambda_i^2\right]$	$x = 0$;p times $= \frac{r\pm\sqrt{r^2+4[(p-r)^2+r^2]}}{2}$ $= 2\lambda_i, -\lambda_i; \lambda_i \neq r$
3	$\begin{bmatrix} A & \overline{A}+I \\ \overline{A}+I & A \end{bmatrix}$	$\prod_{i=1}^{p} \left[(x-\lambda_i)^2 - \{\lambda_i - P(\lambda_i)\}^2\right]$	$x = p, 2r - p$ $= 2\lambda_i; \lambda_i \neq r$ $= 0;$ $p-1$ times
4	$\begin{bmatrix} A & \overline{A}+I & J_{p\times k} \\ \overline{A}+I & 0_p & 0_{p\times k} \\ J_{k\times p} & 0_{k\times p} & 0_k \end{bmatrix}$	$x^k \prod_{i=1}^{p} [x(x-\lambda_i) - kJ] - [J-A]^2$	$x = 0;$ k times $= \frac{r\pm\sqrt{r^2+4[pk+(p-r)^2]}}{2}$ $= \frac{1\pm\sqrt{5}}{2}\lambda_i; \lambda_i \neq r$
5	$\begin{bmatrix} A & \overline{A}+I & \overline{A}+I \\ \overline{A}+I & 0_p & 0_p \\ \overline{A}+I & 0_p & 0_p \end{bmatrix}$	$x^k \prod_{i=1}^{p} \left\{x(x-\lambda_i) - 2[J-A]^2\right\}$	$x = 0;$ p times $= \frac{r\pm\sqrt{r^2+8(p-r)^2}}{2}$ $= 2\lambda_i, -\lambda_i$; $\lambda_i \neq r$
6	$\begin{bmatrix} 0 & A & \overline{A}+I \\ A & 0 & 0 \\ \overline{A}+I & 0 & 0 \end{bmatrix}$	$x^p \prod_{i=1}^{p} \left\{x^2 - [J-A]^2 - A^2\right\}$	$x = 0;$ p times $= \pm\sqrt{r^2 + (p-r)^2}$ $= \pm\sqrt{2}\lambda_i$; $\lambda_i \neq r$
7	$\begin{bmatrix} 0 & \overline{A}+I & J_{p\times k} \\ \overline{A}+I & 0 & 0 \\ J_{k\times p} & 0 & 0 \end{bmatrix}$	$x^k \prod_{i=1}^{p} \left\{x^2 - kJ - (J-A)^2\right\}$	$x = 0;$ k times $= \pm\sqrt{pk + (p-r)^2}$ $= \pm\lambda_i$; $\lambda_i \neq r$

where A, J are, respectively, the adjacency matrix of G and the all one matrix of order p and $J = P(\lambda_i)$ as given by lemma 6.

Operation 10. Introduce p isolated vertices on $U = \{u_1, u_2, \ldots, u_p\}$. Make u_i adjacent to vertices in $\overline{N[v_i]}$.

Operation 11. Introduce p isolated vertices on $U = \{u_1, u_2, \ldots, u_p\}$. Make u_i adjacent to vertices in $\overline{N(v_i)}$.

Operation 12. Introduce p isolated vertices on $U = \{u_1, u_2, \ldots, u_p\}$. Make u_i adjacent to v_i for each i.

Operation 13. Take one copy of G on $U = \{u_1, u_2, \ldots, u_p\}$ and a set $W = \{w_1, w_2, \ldots, w_p\}$ of p isolated vertices. Now join u_i to v_i and w_i to both u_i and v_i for each i.

Operation 14. Introduce p isolated vertices on $U = \{u_1, u_2, \ldots, u_p\}$. Now join u_i to all vertices of G except v_i for each i.

Operation 15. Take a copy of $\overline{G}$ on $U = \{u_1, u_2, \ldots, u_p\}$. Now join u_i to all vertices in $\overline{N[v_i]}$ for each i.

Theorem 4. Let G be an r regular graph on $V(G) = \{v_1, v_2, \ldots, v_p\}$ with spectrum $\{\lambda_1 = r, \lambda_2, \ldots, \lambda_p\}$ and $F_i s$ be the graphs as described above. Then the spectrum of F_i and its complement, $i = 1, 2, \ldots, 8$ are as follows.

i	Spectrum of F_i	Spectrum of $\overline{F_i}$
1	$\left\{\begin{array}{l} \frac{(p-1)\pm\sqrt{(p-2r-1)^2+4}}{2}; \\ \frac{-1\pm\sqrt{1+4(\lambda_i^2+\lambda_i+1)}}{2}; \lambda_i \neq r \end{array}\right\}$	$\left\{\begin{array}{l} \frac{(p-1)\pm\sqrt{(p-1)^2+4\left[(p-1)^2-(p-r-1)r\right]}}{2}; \\ \frac{-1\pm\sqrt{1+4\left[\lambda_i^2-\lambda_i+1\right]}}{2}; \lambda_i \neq r \end{array}\right\}$
2	$\left\{\begin{array}{l} p-1; \\ 2r+1-p \\ -1, (p-1) \text{ times} \\ 2\lambda_i+1; \lambda_i \neq r \end{array}\right\}$	$\left\{\begin{array}{l} p \\ p-2r-2 \\ 0, (p-1) \text{ times} \\ -2(\lambda_i+1); \lambda_i \neq r \end{array}\right\}$
3	$\left\{\begin{array}{l} \frac{r\pm\sqrt{r^2+4(p-r-1)^2}}{2} \\ \frac{\lambda_i\pm\sqrt{5\lambda_i^2+8\lambda_i+4}}{2}; \lambda_i \neq r \end{array}\right\}$	$\left\{\begin{array}{l} \frac{2(p-1)-r\pm\sqrt{r^2+4(r+1)^2}}{2} \\ \frac{-(2+\lambda_i)\pm\sqrt{5\lambda_i^2+8\lambda_i+4}}{2} \end{array}\right\}$
4	$\left\{\begin{array}{l} \frac{r\pm\sqrt{r^2+4(p-r)^2}}{2} \\ \frac{1\pm\sqrt{5}}{2}\lambda_i; \lambda_i \neq r \end{array}\right\}$	$\left\{\begin{array}{l} \frac{2(p-1)-r(1\pm\sqrt{5})}{2} \\ \frac{\left(1\pm\sqrt{5}\right)\lambda_i-2}{2}; \lambda_i \neq r \end{array}\right\}$
5	$\frac{\lambda_i\pm\sqrt{\lambda_i^2+4}}{2}$	$\left\{\begin{array}{l} \frac{2(p-1)-r\pm\sqrt{r^2+4(p-1)^2}}{2} \\ \frac{-(\lambda_i+2)\pm\sqrt{\lambda_i^2+4}}{2}; \lambda_i \neq r \end{array}\right\}$
6	$\left.\begin{array}{l} \lambda_i-1 \\ \frac{\lambda_i+1\pm\sqrt{(\lambda_i+1)^2+8}}{2} \end{array}\right\}$	$\left\{\begin{array}{l} \frac{3(p-1)-r\pm\sqrt{[3(p-1)-r]^2+4r(p-1)}}{2} \\ -\lambda_i \\ \frac{-(3+\lambda_i)\pm\sqrt{(3+\lambda_i)^2-4\lambda_i}}{2}; \lambda_i \neq r \end{array}\right.$
7	$\left.\begin{array}{l} \frac{r\pm\sqrt{r^2+4(p-1)^2}}{2} \\ \frac{\lambda_i\pm\sqrt{\lambda_i^2+4}}{2}; \lambda_i \neq r \end{array}\right\}$	$\left\{\begin{array}{l} \frac{2(p-1)-r\pm\sqrt{r^2+4}}{2} \\ \frac{-(2+\lambda_i)\pm\sqrt{\lambda_i^2+4}}{2}; \lambda_i \neq r \end{array}\right.$
8	$\left.\begin{array}{l} \frac{p-1\pm\sqrt{(p-1)^2+4(p-r-1)(p-2r-1)}}{2} \\ \frac{-1\pm\sqrt{1+4(1+\lambda_i)(1+2\lambda_i)}}{2} \end{array}\right\}$	$\left\{\begin{array}{l} \frac{p-2r-1\pm\sqrt{(p-2r-1)^2-4r(p-r-1)+4(r+1)^2}}{2} \\ \frac{-1\pm\sqrt{1+4(1+\lambda_i)(1+2\lambda_i)}}{2}; \lambda_i \neq r \end{array}\right\}.$

Proof. Table 2 gives the adjacency matrices of the graphs F_i and its complement under each of the operation for $i = 1, \ldots, 8$.

Table 2
Adjacency matrix of F_i and its complement.

i	Adjacency matrix of F_i	Adjacency matrix of $\overline{F_i}$
1	$\begin{bmatrix} A & I \\ I & \overline{A} \end{bmatrix}$	$\begin{bmatrix} \overline{A} & J-I \\ J-I & A \end{bmatrix}$
2	$\begin{bmatrix} A & \overline{A} \\ \overline{A} & A \end{bmatrix}$	$\begin{bmatrix} \overline{A} & A+I \\ A+I & \overline{A} \end{bmatrix}$
3	$\begin{bmatrix} A & \overline{A} \\ \overline{A} & 0_p \end{bmatrix}$	$\begin{bmatrix} \overline{A} & A+I \\ A+I & J-I \end{bmatrix}$
4	$\begin{bmatrix} A & \overline{A}+I \\ \overline{A}+I & 0_p \end{bmatrix}$	$\begin{bmatrix} \overline{A} & A \\ A & J-I \end{bmatrix}$
5	$\begin{bmatrix} A & I \\ I & 0 \end{bmatrix}$	$\begin{bmatrix} \overline{A} & J-I \\ J-I & J-I \end{bmatrix}$
6	$\begin{bmatrix} A & I & I \\ I & A & I \\ I & I & 0 \end{bmatrix}$	$\begin{bmatrix} \overline{A} & J-I & J-I \\ J-I & \overline{A} & J-I \\ J-I & J-I & J-I \end{bmatrix}$
7	$\begin{bmatrix} A & J-I \\ J-I & 0 \end{bmatrix}$	$\begin{bmatrix} \overline{A} & I \\ I & J-I \end{bmatrix}$
8	$\begin{bmatrix} A & \overline{A} \\ \overline{A} & \overline{A} \end{bmatrix}$	$\begin{bmatrix} \overline{A} & A+I \\ A+I & A \end{bmatrix}$

Now the theorem follows from table 3, which gives the characteristic polynomial of F_i and $\overline{F_i}$ for $i = 1, 2, \ldots, 8$. □

Table 3
Characteristic polynomial of F_i and its complement.

i	Ch polynomial of F_i	Ch polynomial of $\overline{F_i}$
1	$\prod_{i=1}^{p} \{[x+1+\lambda_i - J][x-\lambda_i] - 1\}$	$\prod_{i=1}^{p} \left\{[x-(J-1-\lambda_i)][x-\lambda_i] - (J-I)^2\right\}$
2	$[x-(p-1)](x+1)^{p-1} \prod_{i=1}^{p} [x+J-2\lambda_i - 1]$	$\prod_{i=1}^{p} \left\langle \{x-[J-I-\lambda_i]\}^2 - [\lambda_i+1]^2 \right\rangle$
3	$\prod_{i=1}^{p} \left[x^2 - \lambda_i x - (J-I-\lambda_i)^2\right]$	$\prod_{i=1}^{p} \{[x-(J-I-\lambda_i)][x-(J-I)] - (\lambda_i+1)^2\}$
4	$\prod_{i=1}^{p} \left[x^2 - \lambda_i x - (J-\lambda_i)^2\right]$	$\prod_{i=1}^{p} \left\{[x-(J-I-\lambda_i)][x-(J-I)] - \lambda_i^2\right\}$
5	$\prod_{i=1}^{p} \left[x^2 - \lambda_i x - 1\right]$	$\prod_{i=1}^{p} \left[x^2 - \{2(J-I) - \lambda_i\}x - \lambda_i (J-I)\right]$
6	$\prod_{i=1}^{p} [x-(\lambda_i - 1)]\left[x^2 - (\lambda_i+1)x - 2\right]$	$\prod_{i=}^{p} (x+\lambda_i) \left[\begin{matrix} x^2 - \{3(J-I) - \lambda_i\}x \\ -\lambda_i (J-I) \end{matrix}\right]$

Table 3
Continued.

i	Ch polynomial of F_i	Ch polynomial of $\overline{F_i}$
7	$\prod_{i=1}^{p} x\,(x-\lambda_i)-(J-I)^2$	$\prod_{i=}^{p}[\{x-(J-I-\lambda_i)\}\{x-J+I\}-1]$
8	$\prod_{i=1}^{p}\left\{(x-\lambda_i)\,(x-J+I+\lambda_i)-(J-I-\lambda_i)^2\right\}$	$\prod_{i=1}^{p}[(x-J+I+\lambda_i)\,(x-\lambda_i)-(1+\lambda_i)^2]$

Where $J = P(\lambda_i)$ as given by lemma 6.

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