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EQUIENERGETIC SELF-COMPLEMENTARY GRAPHS

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Abstract. In this paper equienergetic self-complementary graphs on p vertices for every $p = 4k$, $k \geqslant 2$ and $p = 24t + 1$, $t \geqslant 3$ are constructed.

Keywords: spectrum, self-complementary graph, energy of graphs

MSC 2010: 05C50

1. Introduction

Let G be a graph with $|V(G)| = p$ and let A be an adjacency matrix of G. The eigenvalues of A are called the eigenvalues of G and form the spectrum of G denoted by spec(G) [4]. The energy [3] of G, $E(G)$ is the sum of the absolute values of its eigenvalues. The properties of $E(G)$ are discussed in detail in [7], [8], [9]. Two non-isomorphic graphs with identical spectrum are called cospectral and two non-cospectral graphs with the same energy are called equienergetic. In [2] and [5], a pair of equienergetic graphs on p vertices where $p \equiv 0 \pmod 4$ and $p \equiv 0 \pmod 5$ are constructed respectively. In [10] we have extended the same to $p = 6, 14, 18$ and to every $p \geqslant 20$. In [12] two classes of equienergetic regular graphs have been obtained and in [11], the energies of some non-regular graphs are studied.

In this paper, we provide a construction of equienergetic self-complementary graphs for every $p = 4k$, $k \geqslant 2$ and $p = 24t + 1$, $t \geqslant 3$. The energies of some special classes of self-complementary graphs are also discussed.

All graph theoretic terminology are from [1], [4].

We use the following lemmas in this paper.

Lemma 1 [4]. *Let G be a graph with an adjacency matrix A and* $\operatorname{spec}(G) = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$. *Then* $\det A = \prod_{i=1}^{p} \lambda_i$. *Also for any polynomial $P(x)P(\lambda)$ is an eigenvalue of $P(A)$ and hence* $\det P(A) = \prod_{i=1}^{p} P(\lambda_i)$.

Lemma 2 [4]. *Let M, N, P and Q be matrices with M invertible. Let* $S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$. *Then $|S| = |M||Q - PM^{-1}N|$ and if M and P commute then $|S| = |MQ - PN|$ where the symbol $|\cdot|$ denotes determinant.*

Lemma 2 [12]. *Let G be an r-regular connected graph, $r \geqslant 3$ with* $\operatorname{spec}(G) = \{r, \lambda_2, \ldots, \lambda_p\}$. *Then*

$$\operatorname{spec}(L^2(G)) = \begin{pmatrix} 4r-6 & \lambda_2 + 3r - 6 & \ldots & \lambda_p + 3r - 6 & 2r-6 & -2 \\ 1 & 1 & \ldots & 1 & \frac{1}{2}p(r-2) & \frac{1}{2}pr(r-2) \end{pmatrix},$$

$E(L^2(G)) = 2pr(r-2)$ *and* $E(\overline{L^2(G)}) = (pr-4)(2r-3) - 2$.

Lemma 4 [4]. *Let G be an r-regular connected graph on p vertices with A as an adjacency matrix and $r = \lambda_1, \lambda_2, \ldots, \lambda_m$ as the distinct eigenvalues. Then there exists a polynomial $P(x)$ such that $P(A) = J$ where J is the all one square matrix of order p and $P(x)$ is given by*

$$P(x) = p \times \frac{(x-\lambda_2)(x-\lambda_3)\ldots(x-\lambda_m)}{(r-\lambda_2)(r-\lambda_3)\ldots(r-\lambda_m)},$$

so that $P(r) = p$ and $P(\lambda_i) = 0$, for all $\lambda_i \neq r$.

Let G be an r-regular connected graph. Then the following constructions [6] result in self-complementary graphs H_i, $i = 1$ to 4.

Construction 1. *H_1: Replace each of the end vertices of P_4, the path on 4 vertices, by a copy of G and each of the internal vertices by a copy of $\overline{G}$. Join the vertices of these graphs by all possible edges whenever the corresponding vertices of P_4 are adjacent.*

Construction 2. *H_2: Replace each of the end vertices of P_4, the path on 4 vertices, by a copy of $\overline{G}$ and each of the internal vertices by a copy of G. Join the vertices of these graphs by all possible edges whenever the corresponding vertices of P_4 are adjacent.*

Construction 3. *H_3: Replace each of the end vertices of the non-regular self-complementary graph F on 5 vertices by a copy of $\overline{G}$, each of the vertices of degree 3 by a copy of G and the vertex of degree 2 by K_1. Join the vertices of these graphs by all possible edges whenever the corresponding vertices of F are adjacent.*

Construction 4. *H_4: Consider the regular self-complementary graph $C_5 = v_1v_2v_3v_4v_5v_1$, the cycle on 5 vertices. Replace the vertices v_1 and v_5 by a copy of $\overline{G}$, v_2 and v_4 by a copy of G and v_3 by K_1. Join the vertices of these graphs by all possible edges whenever the corresponding vertices of C_5 are adjacent.*

Note. For all non self-complementary graphs G, Constructions 1 and 2 yield non-isomorphic graphs and for any graph G, $H_1(G) = H_2(\overline{G})$.

2. Equienergetic self-complementary graphs

In this section, we construct a pair of equienergetic self complementary graphs, first for $p = 4k$, $k \geqslant 2$, and then for $p = 24t + 1$, $t \geqslant 3$.

Theorem 1. *Let G be an r-regular connected graph on p vertices with* $\operatorname{spec}(G) = \{r, \lambda_2, \ldots, \lambda_p\}$ *and H_1 be the self-complementary graph obtained by Construction 1. Then* $E(H_1) = 2[E(G) + E(\overline{G}) - (p-1)] + \sqrt{(2p-1)^2 + 4\{(p-r)^2 + r\}} + \sqrt{1 + 4(p^2 + r + r^2)}$.

P r o o f. Let G be an r-regular connected graph on p vertices with an adjacency matrix A, $\operatorname{spec}(G) = \{r, \lambda_2, \ldots, \lambda_p\}$ and H_1 be the self-complementary graph obtained by Construction 1. Then the adjacency matrix of H_1 is $\begin{bmatrix} A & J & 0 & 0 \\ J & \bar{A} & J & 0 \\ 0 & J & \bar{A} & J \\ 0 & 0 & J & A \end{bmatrix}$, so that the characteristic equation of H_1 is

$$\begin{vmatrix} \lambda I - A & -J & 0 & 0 \\ -J & \lambda I - \bar{A} & -J & 0 \\ 0 & -J & \lambda I - \bar{A} & -J \\ 0 & 0 & -J & \lambda I - A \end{vmatrix} = 0,$$

that is,

$$\begin{vmatrix} -J & \lambda I - \bar{A} & 0 & -J \\ \lambda I - \bar{A} & -J & -J & 0 \\ -J & 0 & \lambda I - A & 0 \\ 0 & -J & 0 & \lambda I - A \end{vmatrix} = 0,$$

by a sequence of elementary transformations.

But, the last expression by virtue of Lemma 2 is

$$|J^2(\lambda I - A)^2 - [(\lambda I - A)(\lambda I - \bar{A}) - J^2]^2| = 0,$$

and so $\prod_{i=1}^{p}\{\langle P(\lambda_i)\rangle^2(\lambda-\lambda_i)^2 - [(\lambda-\lambda_i)(\lambda - P(\lambda_i)+1+\lambda_i) - \langle P(\lambda_i)\rangle^2]^2\} = 0$ by Lemmas 1 and 4.

Now, corresponding to the eigenvalue r of G, the eigenvalues of H_1 are given by

$$\{p^2(\lambda-r)^2 - [(\lambda-r)(\lambda-p+1+r)-p^2]^2\} = 0$$

by Lemmas 1 and 4. That is,

$$[\lambda^2+\lambda-(r^2+r+p^2)][\lambda^2-(2p-1)\lambda-\{(p-r)^2+r\}] = 0$$

So

$$\lambda = \frac{-1\pm\sqrt{1+4(p^2+r+r^2)}}{2};\ \frac{2p-1\pm\sqrt{(2p-1)^2+4\{(p-r)^2+r\}}}{2}.$$

The remaining eigenvalues of H_1 satisfy $\prod_{i=2}^{p}[(\lambda-\lambda_i)(\lambda+1+\lambda_i)]^2 = 0$. Hence,

$$\operatorname{spec}(H_1) = \begin{pmatrix} \frac{-1\pm\sqrt{1+4(p^2+r+r^2)}}{2} & \frac{2p-1\pm\sqrt{(2p-1)^2+4\{(p-r)^2+r\}}}{2} & \underset{i=2,\dots,p}{\lambda_i} & \underset{i=2,\dots,p}{-1-\lambda_i} \\ \text{each once} & \text{each once} & \text{each twice} & \text{each twice} \end{pmatrix}.$$

Now, the expression for $E(H_1)$ follows. □

Theorem 2. *Let G be an r-regular connected graph on p vertices with* $\operatorname{spec}(G) = \{r, \lambda_2, \dots, \lambda_p\}$ *and H_2 be the self-complementary graph obtained by Construction 2. Then* $E(H_2) = 2[E(G)+E(\overline{G})-(p-1)] + \sqrt{(2p-1)^2+4\{(p-r)^2+r\}} + \sqrt{1+4(p^2+r+r^2)}$.

P r o o f. Let A be the adjacency matrix of G. Then the adjacency matrix of H_2 is

$$\begin{bmatrix} \bar{A} & J & 0 & 0 \\ J & A & J & 0 \\ 0 & J & A & J \\ 0 & 0 & J & \bar{A} \end{bmatrix}.$$

By a similar computation as in Theorem 1 in which A is replaced by $\bar{A}$, we get the characteristic equation of H_2 as

$$\prod_{i=1}^{p}\{\langle P(\lambda_i)\rangle^2(\lambda - P(\lambda_i) + \lambda_i + 1)^2 \\ - [(\lambda - \lambda_i)(\lambda - P(\lambda_i) + 1 + \lambda_i) - \langle P(\lambda_i)\rangle^2]^2\} = 0,$$

by Lemmas 1, 2 and 4.

Hence

$$\operatorname{spec}(H_2) = \begin{pmatrix} \frac{2p-1\pm\sqrt{1+4(p^2+r+r^2)}}{2} & \frac{-1\pm\sqrt{(2p-1)^2+4\{(p-r)^2+r\}}}{2} & \underset{i=2,\dots,p}{\lambda_i} & \underset{i=2,\dots,p}{-1-\lambda_i} \\ \text{each once} & \text{each once} & \text{each twice} & \text{each twice} \end{pmatrix}.$$

Now, the expression for $E(H_2)$ follows. □

Corollary 1.

1. *If* $G = K_p$, *then* $E(H_1) = E(H_2) = 2(p-1) + \sqrt{1+4p^2} + \sqrt{8p^2-4p+1}$.
2. *If* $G = K_{n,n}$, *then* $p = 2n$ *and* $E(H_1) = E(H_2) = 2(2p-3) + \sqrt{5p^2-2p+1} + \sqrt{5p^2+2p+1}$.

Theorem 3. *For every* $p = 4k$, $k \geqslant 2$, *there exists a pair of equienergetic self-complementary graphs.*

P r o o f. Let H_1 and H_2 be the self-complementary graphs obtained from K_k as in Constructions 1 and 2. Then by Theorems 1 and 2, they are equienergetic on $p = 4k$ vertices. □

Theorem 4. *Let* H_3 *be the self-complementary graph obtained from* K_p *by Construction* 3. *Then* $E(H_3) = 2(p-1) + \sqrt{4p^2+1} + \sqrt{8p^2+4p+1}$.

P r o o f. Let A be the adjacency matrix of K_p. Then by Construction 3, the adjacency matrix of H_3 is

$$\begin{bmatrix} \bar{A} & J & 0_{p\times 1} & 0 & 0 \\ J & A & J_{p\times 1} & J & 0 \\ 0_{1\times p} & J_{1\times p} & 0 & J_{1\times p} & 0 \\ 0 & J & J_{p\times 1} & A & J \\ 0 & 0 & 0 & J & \bar{A} \end{bmatrix}.$$

Now, after a sequence of elementary transformations applied to the rows and columns and by Lemma 2, the characteristic equation is

$$\frac{1}{\lambda^{2p-1}}|[\{\lambda(\lambda I - A) - J\}(\lambda I - \bar{A}) - \lambda J^2]^2 - [(\lambda+1)(\lambda I - \bar{A})J]^2| = 0.$$

Since $G = K_p$ is connected and regular, by Lemmas 1 and 4 the characteristic equation of H_3 is

$$\lambda^{2p-1}(\lambda+1)^{2p-2}(\lambda^2+\lambda-p^2)[\lambda^2-(2p-1)\lambda-p(p+2)]=0.$$

Hence

$$\operatorname{spec}(H_3)=\begin{pmatrix} \frac{-1\pm\sqrt{4p^2+1}}{2} & \frac{2p-1\pm\sqrt{8p^2+4p+1}}{2} & -1 & 0 \\ \text{each once} & \text{each once} & \text{each } (2p-2) \text{ times} & \text{each } (2p-2) \text{ times} \end{pmatrix}.$$

Now, the expression for $E(H_3)$ follows. □

Theorem 5. *Let H_4 be the self-complementary graph obtained from K_p by Construction 4. Then $E(H_4) = 2(2p-1) + \sqrt{4p+1} + \sqrt{8p^2-4p+1}$.*

Proof. Let A be the adjacency matrix of K_p. Then by Construction 4, the adjacency matrix of H_4 is

$$\begin{bmatrix} \bar{A} & J & 0_{p\times 1} & 0 & J \\ J & A & J_{p\times 1} & 0 & 0 \\ 0_{1\times p} & J_{1\times p} & 0_{1\times 1} & J_{1\times p} & 0 \\ 0 & 0 & J_{p\times 1} & A & J \\ J & 0 & 0 & J & \bar{A} \end{bmatrix}.$$

Now, after a sequence of elementary transformations applied to the rows and columns and by Lemma 2, the characteristic equation is

$$\frac{1}{\lambda^{2p-1}}|[\{\lambda(\lambda I-A)-J\}^2+(\lambda-1)J^2][(\lambda-1)J^2+(\lambda I-\bar{A})^2] - \lambda J^2[\lambda(\lambda I-A)-J+\lambda I-\bar{A}]^2|=0.$$

Since $G = K_p$ is connected and regular, by Lemma 4 the characteristic equation of H_4 is

$$\lambda^{(2p-2)}(\lambda+1)^{(2p-2)}(\lambda-2p)(\lambda^2+\lambda-p)(\lambda^2+\lambda-2p^2+p)=0.$$

Hence

$$\operatorname{spec}(H_4)=\begin{pmatrix} 2p & \frac{-1\pm\sqrt{4p+1}}{2} & \frac{2p-1\pm\sqrt{8p^2-4p+1}}{2} & -1 & 0 \\ \text{each once} & \text{each once} & \text{each once} & \text{each } (2p-2) \text{ times} & \text{each } (2p-2) \text{ times} \end{pmatrix}.$$

Now, the expression for $E(H_4)$ follows. □

Corollary 2. *Let G be a connected r-regular graph on p vertices with* $\operatorname{spec}(G) = \{r, \lambda_2, \lambda_3, \ldots, \lambda_p\}$ *and H be the self-complementary graph obtained as in Construction 4. Then*

$$E(H) = 2[E(G) + E(\overline{G}) - (p-1)] + \sqrt{1 + 4(p^2 + r + r^2)} + T$$

where T is the sum of absolute values of roots of the cubic

$$x^3 - (2p-1)x^2 - [p^2 - 2p(r-1) + r(r+1)]x + 2p(2p - r - 1) = 0.$$

Lemma 5. *There exists a pair of non-cospectral cubic graphs on $2t$ vertices, for every $t \geqslant 3$.*

P r o o f. Let G_1 and G_2 be the non-cospectral cubic graphs on six vertices labelled as $\{v_j\}$ and $\{u_j\}$, $j = 1$ to 6, respectively.

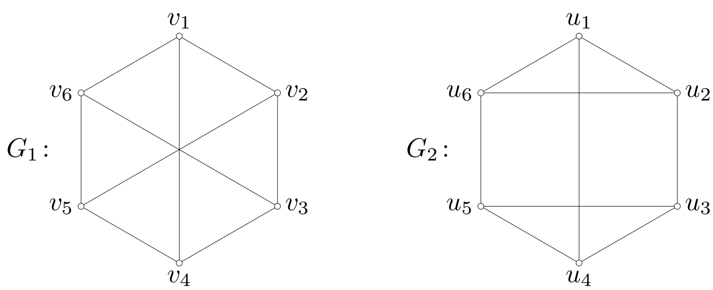

Figure 1. The graphs G_1 and G_2

Now replacing v_1 and u_1 in G_1 and G_2 by a triangle each we get two cubic graphs $\mathscr{H}_1$ and $\mathscr{H}_2$ on eight vertices containing one and two triangles respectively as shown in Figure 2. Since the number of triangles in a graph is the negative of half the coefficient of λ^{p-3} in its characteristic polynomial [4], $\mathscr{H}_1$ and $\mathscr{H}_2$ are non-cospectral.

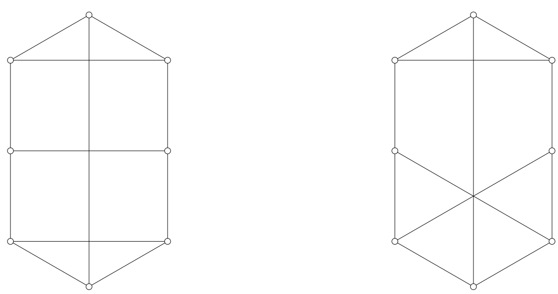

Figure 2. The graphs $\mathscr{H}_1$ and $\mathscr{H}_2$

Replacing any vertex in the newly formed triangle in $\mathscr{H}_1$ and $\mathscr{H}_2$ by a triangle we get two cubic graphs on ten vertices which are non co-spectral. Repeating this process $(t-3)$ times, we get two cubic graphs on $2t$ vertices containing one and two triangles respectively. Hence they are non cospectral. $\square$

Theorem 6. *For every $p = 24t + 1$, $t \geqslant 3$, there exists a pair of equienergetic self-complementary graphs.*

Proof. Let G_1 and G_2 be the two non co-spectral cubic graphs on $2t$ vertices given by Lemma 5. Let F_1 and F_2 respectively denote their second iterated line graphs. Then F_1 and F_2 have $6t$ vertices each and are $6-$regular with $E(F_1) = E(F_2) = 12t$ and $E(\overline{F_1}) = E(\overline{F_2}) = 3(6t-4) - 2$ by Lemma 3. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be the self-complementary graphs obtained from F_1 and F_2 by Construction 4. Then $\mathscr{F}_1$ and $\mathscr{F}_2$ are on $p = 24t + 1$ vertices and by Corollary 2, $E(\mathscr{F}_1) = E(\mathscr{F}_2) = 2(24t-13) + \sqrt{169 + 144t^2} + T$ where T is the sum of the absolute values of the roots of the cubic $x^3 - (12t-1)x^2 - 6(6t^2 - 10t + 7)x + 12t(12t-7) = 0$. $\square$

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