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THE SPECTRUM OF NEIGHBORHOOD CORONA OF GRAPHS

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Abstract. Given two graphs G_1 with vertices $\{v_1, v_2, \ldots, v_n\}$ and G_2, the neighbourhood corona, $G_1 \bigstar G_2$ is the graph obtained by taking n copies of G_2 and for each i, making all vertices in the i^{th} copy of G_2 adjacent with the neighbours of v_i, $i = 1, 2, \ldots, n$. In this paper a complete description of the spectrum and eigenvectors of $G_1 \bigstar G_2$ is given when G_2 is regular, thus adding to the class of graphs whose spectrum is completely known.

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix of G is an $n \times n$ matrix denoted by $A(G) = [a_{ij}]$ and is defined as $a_{ij} = 1$ if v_i and v_j are adjacent in G, 0 otherwise. The spectrum of G is defined as $sp(G) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ where $\lambda_1 \geq \lambda_2 \geq \lambda_n$ are the eigenvalues $A(G)$. The Laplacian matrix of G, denoted by $L(G)$ is defined as $D(G) - A(G)$ where $D(G)$ is the diagonal degree matrix of G. The Laplacian spectrum of G is defined as

$$S(G) = \{\theta_1, \theta_2, \ldots, \theta_n\}$$

where $0 = \theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_n$ are the eigenvalues of $L(G)$. We refer $\lambda_1(G)$ and $\theta_n(G)$ the spectral radius and Laplacian spectral radius, respectively. A plethora of papers have been available on works related to spectrum and Laplacian spectrum of a graph. See [2, 6, 7] and the references cited therein.

The corona of two graphs is defined in [4] and there have been some results on the corona of two graphs [3]. The complete information about the spectrum of the corona

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of two graphs G, H in terms of the spectrum of G, H are given in [1]. A new variance of corona is defined in [5] and discussed its spectrum and the number of spanning trees.

In this paper we define another variation of corona of two graphs and discuss its spectrum , thus adding to the class of graphs whose spectrum is completely known. The discussion in subsequent sections are based upon the following definition.

Definition 1.1. Let G_1 and G_2 be two graphs on n_1 and n_2 vertices, m_1 and m_2 edges respectively. Then the *neighborhood corona*, $G_1 \bigstar G_2$ is the graph obtained by taking n_1 copies of G_2 and for each i, making all vertices in the i^{th} copy of G_2 adjacent with the neighbors of v_i, $i = 1, 2, \ldots, n$.

The neighborhood corona $G_1 \bigstar G_2$ of G_1 and G_2 has $n_1 + n_1 n_2$ vertices and $m_1(2n_2 + 1) + n_1 m_2$ edges and when $G_2 = K_1$, $G_1 \bigstar G_2$ is the splitting graph defined in [8]. Note that in general this operation is not commutative.

Example 1.1. The following figure illustrates Definition 1.1.

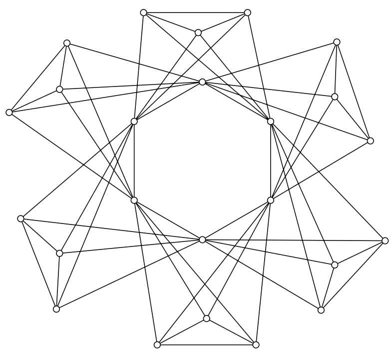

FIGURE 1. $C_6 \bigstar K_2$

Throughout this paper we consider only simple graphs. In this paper, we give a complete description of the eigenvalues and the corresponding eigenvectors of the adjacency matrix of $G_1 \bigstar G_2$ when G_2 is regular.

Let $A = (a_{ij}), B$ be matrices. Then the Kronecker product of A and B is defined in [2] as the partition matrix $(a_{ij}B)$ and is denoted by $A \otimes B$. The row vector of size n with all entries equal to one is denoted by j_n and the identity matrix of order n is denoted by I_n.

2. The spectrum of $G_1 \bigstar G_2$

In this section we obtain the spectrum of $G_1 \bigstar G_2$ when G_2 is regular.

Theorem 2.1. *Let G_1 be a graph with spectrum $\{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n_1}\}$ and G_2, a $t-$regular graph with spectrum $\{\eta_1 = t \geq \eta_2 \geq \ldots \geq \eta_{n_2}\}$. Then the spectrum $\sigma(G)$ of $G_1 \bigstar G_2$ consists of the numbers*

$$\frac{\mu_i + t + \sqrt{(\mu_i - t)^2 + 4n_2\mu_i^2}}{2}, \quad \frac{\mu_i + t - \sqrt{(\mu_i - t)^2 + 4n_2\mu_i^2}}{2}, \quad i = 1, 2, \ldots, n_1,$$

each with multiplicity one together with η_j, $j = 2, 3, \ldots, n_2$, each with multiplicity n_1.

Proof. Let $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, the vertices in the i^{th} copy of G_2 be $\{u_1^i, u_2^i, \ldots, u_{n_2}^i\}$ and $W_j = \{u_j^1, u_j^2, \ldots, u_j^{n_1}\}$. Then with the vertex partition $V \bigcup W_1 \bigcup W_2 \bigcup \cdots \bigcup W_{n_2}$, the adjacency matrix of $G_1 \bigstar G_2$ can be written in the block form

$$\begin{bmatrix} A(G_1) & j_{n_2} \otimes A(G_1) \\ (j_{n_2} \otimes A(G_1))^T & A(G_2) \otimes I_{n_1} \end{bmatrix}.$$

Let X_i be an eigenvector of $A(G_1)$ corresponding to the eigenvalue μ_i, $i = 1, 2, \ldots, n_1$. Let $\delta_i = \dfrac{\mu_i + t + \sqrt{(\mu_i - t)^2 + 4n_2\mu_i^2}}{2}$ and $\hat{\delta}_i = \dfrac{\mu_i + t - \sqrt{(\mu_i - t)^2 + 4n_2\mu_i^2}}{2}$. Now we shall consider two cases.

Case 1. $\mu_i \neq 0$.

In this case it is easy to observe that $\delta_i \neq t$ and $\hat{\delta}_i \neq -t$.

Now $\Phi_i = \begin{bmatrix} \dfrac{\delta_i - t}{\mu_i} X_i \\ j_{n_2} \otimes X_i \end{bmatrix}$ is an eigenvector of $A(G_1 \bigstar G_2)$ with an eigenvalue δ_i. This is because

$$\begin{aligned} A(G_1 \bigstar G_2) \cdot \Phi_i &= \begin{bmatrix} A(G_1) & j_{n_2} \otimes A(G_1) \\ (j_{n_2} \otimes A(G_1))^T & A(G_2) \otimes I_{n_1} \end{bmatrix} \begin{bmatrix} \dfrac{\delta_i - t}{\mu_i} X_i \\ j_{n_2} \otimes X_i \end{bmatrix} \\ &= \begin{bmatrix} \left(\dfrac{\delta_i - t}{\mu_i} \mu_i + n_2\mu_i \right) X_i \\ \left(\dfrac{\delta_i - t}{\mu_i} \mu_i + t \right) j_{n_2} \otimes X_i \end{bmatrix} \\ &= \begin{bmatrix} ((\delta_i - t) + n_2\mu_i) X_i \\ ((\delta_i - t) + t) j_{n_2} \otimes X_i \end{bmatrix} \\ &= \delta_i \begin{bmatrix} \dfrac{\delta_i - t}{\mu_i} X_i \\ j_{n_2} \otimes X_i \end{bmatrix} = \delta_i \Phi_i. \end{aligned}$$

Similarly, it can be proved that $\hat{\Phi}_i = \begin{bmatrix} \frac{\hat{\delta}_i - t}{\mu_i} X_i \\ j_{n_2} \otimes X_i \end{bmatrix}$ is an eigenvector of $A\left(G_1 \bigstar G_2\right)$ with an eigenvalue $\hat{\delta}_i$.

Case 2. $\mu_i = 0$.

In this case observe that $\delta_i = t$ and $\hat{\delta}_i = 0$.
Let $\Phi_i^0 = \begin{bmatrix} 0 \\ j_{n_2} \otimes X_i \end{bmatrix}$. Now Φ_i^0 is an eigenvector of $A\left(G_1 \bigstar G_2\right)$ with an eigenvalue $t = \frac{\mu_i + t + \sqrt{(\mu_i - t)^2 + 4n_2\mu_i^2}}{2}$, $\mu_i = 0$. This is because

$$\begin{aligned} A\left(G_1 \bigstar G_2\right) \cdot \Phi_i^0 &= \begin{bmatrix} A\left(G_1\right) & j_{n_2} \otimes A\left(G_1\right) \\ \left(j_{n_2} \otimes A\left(G_1\right)\right)^T & A\left(G_2\right) \otimes I_{n_1} \end{bmatrix} \begin{bmatrix} 0 \\ j_{n_2} \otimes X_i \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ t \cdot j_{n_2} \otimes X_i \end{bmatrix} \\ &= t \begin{bmatrix} 0 \\ j_{n_2} \otimes X_i \end{bmatrix} = t \cdot \Phi_i^0. \end{aligned}$$

Similarly, it can be proved that $\hat{\Phi}_i^0 = \begin{bmatrix} X_i \\ 0 \end{bmatrix}$ is an eigenvector of $A\left(G_1 \bigstar G_2\right)$ with an eigenvalue $0 = \frac{\mu_i + t - \sqrt{(\mu_i - t)^2 + 4n_2\mu_i^2}}{2}$, $\mu_i = 0$.

Thus we get $2n_1$ eigenvalues $\frac{\mu_i + t \pm \sqrt{(\mu_i - t)^2 + 4n_2\mu_i^2}}{2}$ of $A\left(G_1 \bigstar G_2\right)$ with eigenvectors described earlier.

Now let Y_j be an eigenvector of $A(G_2)$ with an eigenvalue η_j, $i = 2, 3, \ldots, n_2$. Since G_2 is regular, by Perron-Frobenius theory that Y_j is orthogonal to the all one vector. Let $e_{n_1}^i$ be an $n_1 \times 1$ column vector with i^{th} entry equal to one and all other entries equal to zero. Now $\Psi_j^i = \begin{bmatrix} 0 \\ Y_j \otimes e_{n_1}^i \end{bmatrix}$ is an eigenvector of $A\left(G_1 \bigstar G_2\right)$ with an eigenvalue η_j for each $i = 1, 2, \ldots, n_1$. This is because

$$\begin{aligned} A\left(G_1 \bigstar G_2\right) \cdot \Psi_j^i &= \begin{bmatrix} A\left(G_1\right) & j_{n_2} \otimes A\left(G_1\right) \\ \left(j_{n_2} \otimes A\left(G_1\right)\right)^T & A\left(G_2\right) \otimes I_{n_1} \end{bmatrix} \begin{bmatrix} 0 \\ Y_j \otimes e_{n_1}^i \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \eta_j \cdot Y_j \otimes e_{n_1}^i \end{bmatrix} \\ &= \eta_j \Psi_j^i. \end{aligned}$$

The multiplicity of η_j follows from the observation that Ψ_j^i is an eigenvector for each $i = 1, 2, \ldots, n_1$. Thus we have listed $n_1 n_2 + n_2$ eigenvectors for $A\left(G_1 \bigstar G_2\right)$ and by the very construction, they are all linearly independent and as $A(G_1 \bigstar G_2)$ has a basis consisting of linearly independent eigenvectors, the theorem follows. $\square$

3. The Laplacian spectrum of $G_1 \bigstar G_2$

In this section we obtain the Laplacian spectrum of $G_1 \bigstar G_2$.

Let $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, the vertices in the i^{th} copy of G_2 be $\{u_1^i, u_2^i, \ldots, u_{n_2}^i\}$ and $W_j = \{u_j^1, u_j^2, \ldots, u_j^{n_1}\}$. Then with the vertex partition $V \bigcup W_1 \bigcup W_2 \bigcup \cdots \bigcup W_{n_2}$, we have the following degree relations in $G_1 \bigstar G_2$:

$$\deg_{G_1 \bigstar G_2} v_i = \deg_{G_1} v_i \,(n_2 + 1)$$

$$\deg_{G_1 \bigstar G_2} u_j^i = \deg_{G_2} u_j + \deg_{G_1} v_i, \quad i = 1, 2, \ldots, n_1; \quad j = 1, 2, \ldots, n_2.$$

Thus the degree diagonal matrix $D(G_1 \bigstar G_2)$ can be written in the block form

$$\begin{bmatrix} (n_2+1)\,D(G_1) & j_{n_2} \otimes 0 \\ (j_{n_2} \otimes 0)^T & D(G_2) \otimes I_{n_1} + I_{n_2} \otimes D(G_1) \end{bmatrix}.$$

Let $L(G_1)$ and $L(G_2)$ respectively denote the Laplacian matrices of G_1 and G_2, then the Laplacian matrix, $L(G_1 \bigstar G_2)$ of $G_1 \bigstar G_2$ is the block matrix

$$\begin{bmatrix} (n_2+1)\,L(G_1) & -j_{n_2} \otimes A(G_1) \\ (-j_{n_2} \otimes A(G_1))^T & L(G_2) \otimes I_{n_1} + I_{n_2} \otimes D(G_1) \end{bmatrix}.$$

In what follows, we give a complete description of the Laplacian eigenvalues and eigenvectors of $G_1 \bigstar G_2$ for an r-regular graph G_1 and any graph G_2. Note that if G is an r-regular graph with adjacency spectrum $\{r = \lambda_1 \geq \lambda_2, \ldots, \lambda_n\}$, then its Laplacian spectrum is $\{0 = \theta_1 \leq \theta_2 = r - \lambda_2 \leq \ldots \leq \theta_n = r - \lambda_n\}$.

Theorem 3.1. *Let G_1 be an r-regular graph with adjacency spectrum $\{r = \mu_1 \geq \mu_2, \ldots, \mu_n\}$, $S(G_1) = \{\theta_1, \theta_2, \ldots, \theta_{n_1}\}$ and G_2 be a graph with $S(G_2) = \{\tau_1, \tau_2, \ldots, \tau_{n_2}\}$. Then the Laplacian spectrum of $G_1 \bigstar G_2$ consists of the numbers*

$$\frac{(n_2+1)\,r + \theta_i + \sqrt{((n_2+1)\,r + \theta_i)^2 + 4\theta_i\,(n_2\theta_i - r\,(2n_2+1))}}{2},$$

$$\frac{(n_2+1)\,r + \theta_i - \sqrt{((n_2+1)\,r + \theta_i)^2 + 4\theta_i\,(n_2\theta_i - r\,(2n_2+1))}}{2}, \quad i = 1, 2, \ldots, n_1$$

together with $(\tau_j + r)$ of multiplicity n_1 for each $j = 2, 3, \ldots, n_2$.

Proof. Let X_i be an eigenvector with an eigenvalue μ_i of $A(G_1)$. As G_1 is r-regular, X_i is an eigenvector of $L(G_1)$ with an eigenvalue $\theta_i = r - \mu_i$.

Let

$$\sigma_i = \frac{(n_2+1)\,r + \theta_i + \sqrt{((n_2+1)\,r+\theta_i)^2 + 4\theta_i\,(n_2\theta_i - r\,(2n_2+1))}}{2}$$

and

$$\hat{\sigma}_i = \frac{(n_2+1)\,r + \theta_i - \sqrt{((n_2+1)\,r+\theta_i)^2 + 4\theta_i\,(n_2\theta_i - r\,(2n_2+1))}}{2}.$$

Now we shall consider two cases.

Case 1. $\mu_i \neq 0$ *or in other words* $\theta_i \neq r$.

We observe that $\sigma_i, \hat{\sigma}_i = r$ only when $\theta_i = r$ and hence $\sigma_i, \hat{\sigma}_i \neq r$ when $\mu_i \neq 0$. Now in this case, $\Theta_i = \begin{bmatrix} \dfrac{\sigma_i - r}{\theta_i - r} X_i \\ j_{n_2\times 1} \otimes X_i \end{bmatrix}$ is an eigenvector of $L(G_1 \bigstar G_2)$ with an eigenvalue σ_i.

This is because

$$\begin{aligned} & L(G_1\bigstar G_2)\Theta_i \\ &= \begin{bmatrix} (n_2+1)\,L\,(G_1) & -j_{n_2}\otimes A\,(G_1) \\ (-j_{n_2}\otimes A\,(G_1))^T & L\,(G_2)\otimes I_{n_1} + rI_{n_2}\otimes I_{n_1} \end{bmatrix} \begin{bmatrix} \dfrac{\sigma_i - r}{\theta_i - r} X_i \\ j_{n_2\times 1}\otimes X_i \end{bmatrix} \\ &= \begin{bmatrix} \left((n_2+1)\dfrac{\sigma_i - r}{\theta_i - r}(r-\mu_i) - n_2\mu_i\right) X_i \\ \left(-\left(\dfrac{\sigma_i - r}{\theta_i - r}\right) + r\right) X_i \end{bmatrix} \because L\,(G_2) \perp \; j_{n_2\times 1} \\ &= \sigma_i \begin{bmatrix} \dfrac{\sigma_i - r}{\theta_i - r} X_i \\ j_{n_2\times 1}\otimes X_i \end{bmatrix} \end{aligned}$$

Similarly, it can be proved that $\hat{\Theta}_i = \begin{bmatrix} \dfrac{\hat{\sigma}_i - r}{\theta_i - r} X_i \\ j_{n_2\times 1}\otimes X_i \end{bmatrix}$ is an eigenvector of $L(G_1\bigstar G_2)$ with an eigenvalue $\hat{\sigma}_i$.

Case 2. $\mu_i = 0$ *or in other words* $\theta_i = r$

In this case by applying similar arguments we used in the above discussion it can be showed that $\begin{bmatrix} X_i \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ j_{n_2\times 1}\otimes X_i \end{bmatrix}$ respectively are the eigenvectors with eigenvalues

$$\sigma_i = \frac{(n_2+1)\,r + \theta_i + \sqrt{((n_2+1)\,r+\theta_i)^2 + 4\theta_i\,(n_2\theta_i - r\,(2n_2+1))}}{2} = (n_2+1)r$$

and

$$\hat{\sigma}_i = \frac{(n_2+1)\,r+\theta_i-\sqrt{((n_2+1)\,r+\theta_i)^2+4\theta_i\,(n_2\theta_i-r\,(2n_2+1))}}{2} = r.$$

Now let $Z_1, Z_2, \ldots, Z_{n_2}$ be the set of eigenvectors of $L(G_2)$. Then, as shown in the proof of Theorem 2.1, it can be shown that $\Omega_j^i = \begin{bmatrix} 0 \\ Z_j \otimes e_{n_1}^i \end{bmatrix}$ is an eigenvector of $L(G_1 \bigstar G_2)$ with an eigenvalue $\tau_j + r$ of multiplicity n_1 for $j = 2, 3, \ldots, n_2$. Thus we get $n_1 n_2 + n_2$ eigenvectors for $L\,(G_1 \bigstar G_2)$ and by the very construction, they are all linearly independent. Thus, the theorem is proved. □

3.1. **An application.** As an application of the above results, in this section we obtain an expression for the number of spanning trees of $G_1 \bigstar G_2$ for an r-regular graph G_1 with $S(G_1) = \{0 = \theta_1, \theta_2, \ldots, \theta_{n_1}\}$ and G_2 with $S(G_2) = \{\tau_1, \tau_2, \ldots, \tau_{n_2}\}$. From the Matrix-Tree theorem [2] the number of spanning trees of G_1 is

$$t\,(G_1) = \frac{\theta_2\theta_3\ldots\theta_{n_1}}{n_1}.$$

Theorem 3.2. *Let G_1 be a connected r-regular graph with $S(G_1) = \{0 = \theta_1, \theta_2, \ldots, \theta_{n_1}\}$ and G_2 be any arbitrary graph with $S(G_2) = \{\tau_1, \tau_2, \ldots, \tau_{n_2}\}$. Let $t(G_1)$ be the number of spanning trees of G_1. Then*

$$t\,(G_1 \bigstar G_2) = rt\,(G_1) \prod_{i=2}^{n_1} ((2n_2+1)\,r - n_2\theta_i) \prod_{j=2}^{n_2} (\tau_j + r)^{n_1}.$$

Proof. Using the notations in Theorem 3.1, we have $\sigma_i\hat{\sigma}_i = \theta_i\,((2n_2+1)\,r - n_2\theta_i)$, $\sigma_1 = (n_2+1)\,r; \hat{\sigma}_1 = 0$ for $i = 2, 3, \ldots, n_1$. Thus

$$\begin{aligned} t\,(G_1 \bigstar G_2) &= \frac{(n_2+1)\,r \times \prod_{i=2}^{n_1} \theta_i\,((2n_2+1)\,r - n_2\theta_i) \times \prod_{j=2}^{n_2} (\tau_j + r)^{n_1}}{n_1\,(n_2+1)} \\ &= rt\,(G_1) \prod_{i=2}^{n_1} ((2n_2+1)\,r - n_2\theta_i) \prod_{j=2}^{n_2} (\tau_j + r)^{n_1}. \end{aligned}$$

□

Corollary 3.1.

$$\begin{aligned} &t(K_{n_1} \bigstar K_{n_2}) \\ &= n_1^{n_1-2}\,(n_1+n_2-1)^{n_1(n_2-1)}\,(n_1-1)\,((2n_2+1)\,(n_1-1) - n_1n_2)^{n_1-1}. \end{aligned}$$

Proof. The proof follows from the fact that $\theta_i = n_1$, $i = 2, 3, \ldots, n_1$; $\tau_j = n_2$, $j = 2, 3, \ldots, n_2$; $r = n_1 - 1$ and $t(K_{n_1}) = n_1^{n_1-2}$. □

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