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ScienceDirect
AKCE International Journal of Graphs and Combinatorics 12 (2015) 186–192
www.elsevier.com/locate/akcej

The distance spectrum of corona and cluster of two graphs[☆]

G. Indulal[a,*], Dragan Stevanović[b,c]

[a] Department of Mathematics, St. Aloysius College, Edathua, Alappuzha 689573, India
[b] University of Primorska, Institute Andrej Marusic, Muzejski trg 2, 6000 Koper, Slovenia
[c] Mathematical Institute of the Serbian Academy of Science and Arts, Knez Mihajlova 36, 11000 Belgrade, Serbia

Received 19 January 2015; accepted 2 September 2015
Available online 3 December 2015

Abstract

Let G be a connected graph with a distance matrix D. The D-eigenvalues $\{\mu_1, \mu_2, \ldots, \ldots, \mu_p\}$ of G are the eigenvalues of D and form the distance spectrum or D-spectrum of G. Given two graphs G with vertex set $\{v_1, v_2, \ldots\ldots, v_p\}$ and H, the corona $G \circ H$ is defined as the graph obtained by taking p copies of H and for each i, joining the ith vertex of G to all the vertices in the ith copy of H. Let H be a rooted graph rooted at u. Then the cluster $G\{H\}$ is defined as the graph obtained by taking p copies of H and for each i, joining the ith vertex of G to the root in the ith copy of H. In this paper we describe the distance spectrum of $G \circ H$, for a connected distance regular graph G and any r-regular graph H in terms of the distance spectrum of G and adjacency spectrum of H. We also describe the distance spectrum of $G\{K_n\}$, where G is a connected distance regular graph.

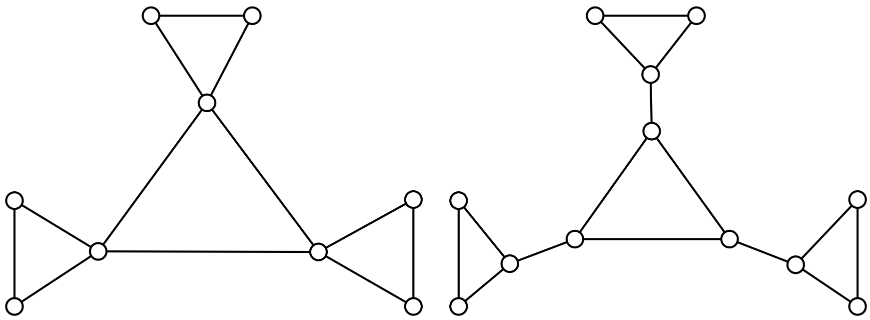
Keywords: Distance spectrum; Corona; Cluster

1. Introduction

The computation of various graph polynomials and the associated spectra has been the topic of many investigations in the recent years. While the problem of computing the characteristic polynomial of adjacency matrix and its spectrum appears to be solved for many large graphs, the related distance polynomial has received much less attention. The idea of distance matrix seems a natural generalization, reflects the structure of the graph in a better way than that of an adjacency matrix. Distance matrix and its spectra have arisen independently from a data communication problem studied by Graham and Pollack in 1971 in which the most important feature is the number of negative eigenvalues of the distance matrix. The distance matrix is more complex than the ordinary adjacency matrix of a graph since the distance matrix is a complete matrix (dense) while the adjacency matrix is more often sparse. Thus the computation of the characteristic polynomial of the distance matrix is computationally a much more difficult problem and, in general,

Peer review under responsibility of Kalasalingam University.
☆ This work was supported by the University Grant Commission of Government of India under the minor research project grant No: MRP(S)-399/08-09/KLMG019/UGC-SWRO.
* Corresponding author.
E-mail address: indulalgopal@gmail.com (G. Indulal).

http://dx.doi.org/10.1016/j.akcej.2015.11.014
0972-8600/



(a) $C_3 \circ K_2$. (b) $C_3\{C_3\}$.

Fig. 1. Corona and cluster of graphs.

there are no simple analytical solutions except those for a few trees [1]. For this reason, distance polynomials of only trees have been studied extensively in the mathematical literature.

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$. The distance matrix $D = D(G)$ of G is defined such that its (i, j)-entry is equal to $d_G(v_i, v_j)$, the distance (=length of the shortest path [2]) between the vertices v_i and v_j of G. The eigenvalues of $D(G)$ are said to be the D-eigenvalues of G and form the D-spectrum of G, denoted by $spec_D(G)$.

The distance spectrum was known earlier only for a few families of graphs, including cycles [3] and complete bipartite graphs [4]. In [5] the distance spectrum of a path P_n and the first distance eigenvector of connected graphs are obtained. In [6] the authors describe the distance spectrum of the join based compositions of regular graphs and in [7] Stevanović generalized this result as the joined union of regular graphs. In [8], we obtain the D-spectrum of the cartesian product of two distance regular graphs and also that of the lexicographic product of a graph with a regular graph. In [9] we have generalized the construction of the graph $K_{n,n+1} \equiv K_{n+1,n}$ of [10] to $G_1 \triangledown G_2 \equiv G_2 \triangledown G_1$ for two regular graphs G_1 and G_2 and obtained its distance spectrum. For some other recent results on D-spectrum of graphs see [11–14].

This work is motivated from [15] in which the authors describe the adjacency spectrum of corona of two graphs and also from a recent work [8] which studied the distance spectrum of some graph compositions. In this paper we obtain the distance spectrum of the corona of a distance regular graph with a regular graph and that of the cluster of a connected distance regular graph with complete graph. All graphs considered in this paper are simple and we follow [16] for spectral graph theoretic terminology. We denote by $spec_A$, the adjacency spectrum of the matrix A.

The considerations in the subsequent sections are based on the applications of the following lemmas:

Lemma 1 ([17]). *Let A be a square matrix with eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$. Then $\det A = \prod_{i=1}^{p} \lambda_i$. In addition, for any polynomial $P(x)$, $P(\lambda)$ is an eigenvalue of $P(A)$.*

Lemma 2 ([12]). *Let D be the distance matrix of a connected distance regular graph G on p vertices and J, the all one matrix of order p. Then D is irreducible and there exists a polynomial $Q(x)$ such that $Q(D) = J$. In this case*

$$Q(x) = p \times \frac{(x - \mu_2)(x - \mu_3) \cdots (x - \mu_g)}{(k - \mu_2)(k - \mu_3) \ldots (k - \mu_g)}$$

where k is the unique sum of each row which is also the greatest simple eigenvalue of D, whereas $\mu_2, \mu_3, \ldots, \mu_g$ are the other distinct eigenvalues of D so that $Q(k) = p$ and $Q(\mu) = 0$ for every eigenvalue μ of D different from k.

Definition 1 ([18]). Given two graphs G with vertex set $\{v_1, v_2, \ldots\ldots, v_p\}$ and H, the corona of G and H is denoted by $G \circ H$ and is defined as the graph obtained by taking p copies of H and for each i, joining the ith vertex of G to all the vertices in the ith copy of H, $i = 1, 2, \ldots\ldots, p$.

Definition 2 ([18]). Let H be a rooted graph rooted at u. Then given a graph G with vertex set $\{v_1, v_2, \ldots\ldots, v_p\}$, the cluster $G\{H\}$ is defined as the graph obtained by taking p copies of H and for each i, joining the ith vertex of G to the root in the ith copy of H.

Example 1. *Let $G = C_3$, the cycle on three vertices and $H = K_2$, the complete graph on two vertices. Then Fig. 1 shows the corona $G \circ H$ and cluster $G\{H\}$.*

2. The distance spectrum of corona of two graphs

In [15] S. Barik et al. obtained the adjacency spectrum of corona of two graphs. In this section we obtain the distance spectrum of the corona of a distance regular graph with a regular graph.

Theorem 1. *Let G be a distance regular graph on p vertices $\{v_1, v_2, \ldots\ldots, v_p\}$ with distance regularity k, a distance matrix D and $spec_D = \{k = \mu_1, \mu_2, \ldots\ldots\ldots, \mu_p\}$. Let H be an r-regular graph on n vertices with an adjacency matrix A and $spec_A = \{r = \lambda_1, \lambda_2, \ldots\ldots, \lambda_n\}$. Then the distance spectrum of $G \circ H$ consists of the following numbers:*

(a) $\frac{n(2p+k)+k-r-2\pm\sqrt{(n(2p+k)+k-r-2)^2+4(np^2+k(r+2))}}{2}$ *each with multiplicity* 1

(b) $\frac{\mu_i(n+1)-r-2\pm\sqrt{(\mu_i(n+1)-r-2)^2+4\mu_i(r+2)}}{2}$ *for each* $\mu_i \in spec_D, i = 2, 3, \ldots, p$

(c) $-\lambda_i - 2$ *with multiplicity* p *for each* $\lambda_i \in spec_A(H), i = 2, 3, \ldots, n$.

Proof. Given that G is a distance regular graph on p vertices $\{v_1, v_2, \ldots\ldots, v_p\}$ with distance regularity k, a distance matrix D and D-spectrum $\{k = \mu_1, \mu_2, \ldots\ldots\ldots, \mu_p\}$. Let the vertex set of the ith copy of H be $U^i = \{u_1^i, u_2^i, \ldots\ldots\ldots, u_n^i\}, i = 1, 2, \ldots\ldots p$.

Let $W_i = \{u_i^1, u_i^2, \ldots\ldots, u_i^p\}$. With this labeling $V(G \circ H) = V(G) \bigcup W_1 \bigcup W_2 \ldots\ldots \bigcup W_n$. Then by the definition of $G \circ H$, its distance matrix F can be written in the form

$$F = \begin{bmatrix} D & J_{1\times n} \otimes (D + J) \\ J_{n\times 1} \otimes (D + J) & J_n \otimes (D + 2(J - I)) + (2(J - I) - A) \otimes I_p \end{bmatrix}.$$

Now as G is distance regular with distance regularity k. Therefore the all one vector $Y = J_{p\times 1}$ is an eigenvector of D corresponding to the eigenvalue k.

Now let

$$\delta_1 = \frac{n(2p + k) + k - r - 2 + \sqrt{(n(2p + k) + k - r - 2)^2 + 4\left(np^2 + k(r + 2)\right)}}{2} \quad \text{and}$$

$$\widehat{\delta_1} = \frac{n(2p + k) + k - r - 2 - \sqrt{(n(2p + k) + k - r - 2)^2 + 4\left(np^2 + k(r + 2)\right)}}{2}.$$

Note that $\delta_1, \widehat{\delta_1} = k$ implies $n = 0$, so that $\delta_1, \widehat{\delta_1}$ are never k.

Claim 1. $\Psi = \begin{bmatrix} \frac{n(k+p)}{\delta_1 - k} Y \\ Y \\ \vdots \\ Y \end{bmatrix}$ *is an eigenvector of F corresponding to the eigenvalue δ_1.*

For

$$F \cdot \Psi = \begin{bmatrix} D & J_{1\times n} \otimes \left(D + J_p\right) \\ J_{n\times 1} \otimes \left(D + J_p\right) & J_n \otimes \left(D + 2\left(J_p - I_p\right)\right) + (2(J_n - I_n) - A) \otimes I_p \end{bmatrix} \begin{bmatrix} \frac{n(k+p)}{\delta_1 - k} Y \\ Y \\ \vdots \\ Y \end{bmatrix}$$

$$= \begin{bmatrix} \frac{n(k+p)}{\delta_1 - k} kY + n(k+p)Y \\ \frac{n(k+p)}{\delta_1 - k}(k+p)Y + \{n(k + 2(p-1)) + 2(n-1) - r\}Y \\ \vdots \\ \frac{n(k+p)}{\delta_1 - k}(k+p)Y + \{n(k + 2(p-1)) + 2(n-1) - r\}Y \end{bmatrix} \quad \text{by Lemmas 1 and 2}$$

$$= \begin{bmatrix} \delta_1 \dfrac{n(k+p)}{\delta_1 - k} Y \\ \delta_1 Y \\ \vdots \\ \delta_1 Y \end{bmatrix}$$

$$= \delta_1 \cdot \Psi.$$

Thus δ_1 is an eigenvalue of F of multiplicity 1. By a similar argument we can see that $\widehat{\delta_1}$ is also an eigenvalue of F with eigenvector $\widehat{\Psi} = \begin{bmatrix} \frac{n(k+p)}{\widehat{\delta_1} - k} Y \\ Y \\ \vdots \\ Y \end{bmatrix}$ of multiplicity 1.

Now consider the eigenvalue $\mu_i \neq k$ of D. Then there are two cases.

Case 1. $\mu_i \neq 0$.

Consider the eigenvalue $\mu_i \neq 0$ of D, $i = 2, 3, \ldots\ldots, t$. Let Y_i be the eigenvector corresponding to μ_i, $i = 2, 3, \ldots\ldots, t$. Then by the theory of Perron–Frobenius, Y_i is orthogonal to Y.

Let

$$\delta_i = \frac{\mu_i (n+1) - r - 2 + \sqrt{(\mu_i (n+1) - r - 2)^2 + 4\mu_i (r+2)}}{2} \quad \text{and}$$

$$\widehat{\delta_i} = \frac{\mu_i (n+1) - r - 2 - \sqrt{(\mu_i (n+1) - r - 2)^2 + 4\mu_i (r+2)}}{2}, \quad i = 2, 3, \ldots, t.$$

Note that $\delta_i, \widehat{\delta_i} = \mu_i$ implies $n = 0$, so that $\delta_i, \widehat{\delta_i}$ are never μ_i, $i = 2, 3, \ldots, t$.

Claim 2. $\Psi_i = \begin{bmatrix} \frac{n\mu_i}{\delta_i - \mu_i} Y_i \\ Y_i \\ \vdots \\ Y_i \end{bmatrix}$ *is an eigenvector of F corresponding to the eigenvalue δ_i, $i = 2, 3, \ldots, t$.*

For

$$F \cdot \Psi_i = \begin{bmatrix} D & J_{1\times n} \otimes (D + J_p) \\ J_{n\times 1} \otimes (D + J_p) & J_n \otimes (D + 2(J_p - I_p)) + (2(J_n - I_n) - A) \otimes I_p \end{bmatrix} \begin{bmatrix} \dfrac{n\mu_i}{\delta_i - \mu_i} Y_i \\ Y_i \\ \vdots \\ Y_i \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{n\mu_i}{\delta_i - \mu_i} \mu_i Y_i + n\mu_i Y_i \\ \dfrac{n\mu_i}{\delta_i - \mu_i} \mu_i Y + \{n(\mu_i - 2) + 2(n-1) - r\} Y_i \\ \vdots \\ \dfrac{n\mu_i}{\delta_i - \mu_i} \mu_i Y + \{n(\mu_i - 2) + 2(n-1) - r\} Y_i \end{bmatrix}$$

$$= \begin{bmatrix} \delta_i \dfrac{n\mu_i}{\delta_i - \mu_i} Y_i \\ \delta_i Y_i \\ \vdots \\ \delta_i Y_i \end{bmatrix}$$

$$= \delta_i \cdot \Psi_i.$$

Thus δ_i is an eigenvalue of F of multiplicity 1. By a similar argument we can see that $\widehat{\delta_i}$ is also an eigenvalue of F with eigenvector $\hat{\Psi}_i = \begin{bmatrix} \frac{n\mu_i}{\hat{\delta}_i - \mu_i} Y_i \\ Y_i \\ \vdots \\ Y_i \end{bmatrix}$.

Case 2. $\mu_i = 0$.

Consider the eigenvalue $\mu_i = 0$ of D and let Z_i be the corresponding eigenvectors, $i = t, t+1, \ldots, p$. Then by the theory of Perron–Frobenius, Z_i is orthogonal to Y.

As argued earlier we can see that $-(r+2)$ and 0 are eigenvalues of F corresponding to the eigenvectors $\begin{bmatrix} 0_{p\times 1} \\ Z_i \\ \vdots \\ Z_i \end{bmatrix}$ and $\begin{bmatrix} Z_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ respectively. Statement (b) in the theorem gives $-(r+2)$ and 0 as eigenvalues when $\mu_i = 0$.

Further since H is r-regular it has r as an eigenvalue of A and eigenvectors corresponding to other eigenvalues are orthogonal to the all one vector $J_{n\times 1}$.

Let X_i be an eigenvector corresponding to the eigenvalue $\lambda_i \neq r$ of A and $e^j_{p\times 1}$ be a $p \times 1$ column vector whose jth entry is 1 and all other entries are zeros.

Claim 3. *For each $j = 1, 2, \ldots, p$, $\phi_i^j = \begin{bmatrix} 0 \cdot J_{p\times 1} \\ X_i \otimes e^j_{p\times 1} \end{bmatrix}$ is an eigenvector of F corresponding to the eigenvalue $-\lambda_i - 2$ for every $i = 2, 3, \ldots, n$.*

For

$$\begin{aligned} F \cdot \phi_i^j &= \begin{bmatrix} D & J_{1\times n} \otimes (D + J_p) \\ J_{n\times 1} \otimes (D + J_p) & J_n \otimes (D + 2(J_p - I_p)) + (2(J_n - I_n) - A) \otimes I_p \end{bmatrix} \begin{bmatrix} 0 \cdot J_{p\times 1} \\ X_i \otimes e^j_{p\times 1} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} + (J_{1\times n} \cdot X_i) \otimes (D + J_p) \cdot e^j_{p\times 1} \\ \mathbf{0} \otimes (D + J_p) \cdot J_{p\times 1} + (J_n \cdot X_i) \otimes \left((D + 2(J_p - I_p))\, e^j_{p\times 1}\right) + ((2(J_n - I_n) - A) \cdot X_i) \otimes \left(I_p \cdot e^j_{p\times 1}\right) \end{bmatrix} \\ &= \begin{bmatrix} 0 \cdot J_{p\times 1} \\ (-\lambda_i - 2)\, X_i \otimes e^j_{p\times 1} \end{bmatrix} \\ &= (-\lambda_i - 2) \cdot \phi_i^j. \end{aligned}$$

Thus for a fixed i, $-\lambda_i - 2$ is an eigenvalue of F with eigenvector ϕ_i^j, for $j = 1, 2, \ldots, p$. Hence the multiplicity of $-\lambda_i - 2$ as an eigenvalue of F is p. Thus we have listed all the $np + p$ eigenvalues of F which completes the proof. □

3. The distance spectrum of cluster of a distance regular graph with complete graph

Let G be a distance regular graph with vertex set $V = \{v_1, v_2, \ldots\ldots, v_p\}$ and a distance matrix D. Also let the vertex set of the ith copy of K_n be $\{u_1^i, u_2^i, \ldots, u_n^i\}$ with root u_1^i. Let $W_j = \{u_j^1, u_j^2, \ldots, u_j^p\}$. With this labeling $V(G\{K_n\}) = V(G) \bigcup W_1 \bigcup W_2 \ldots\ldots \bigcup W_n$.

Then by the definition of the cluster of two graphs, its distance matrix F can be written in the form

$$F = \begin{bmatrix} D & D + J & J_{1\times n-1} \otimes (D + 2J) \\ D + J & D + 2(J - I) & J_{1\times n-1} \otimes (D + 3J - 2I) \\ J_{n-1\times 1} \otimes (D + 2J) & J_{n-1\times 1} \otimes (D + 3J - 2I) & I_{n-1} \otimes (D + 4(J - I)) + (J - I)_{n-1} \otimes (D + 4J - 3I) \end{bmatrix}$$

where J is the all-one matrix and I, the identity matrix of appropriate orders. Now we shall find the distance spectrum of F.

Theorem 2. *Let G be a distance regular graph with distance regularity k, a distance matrix D and distance spectrum $\{k=\mu_1,\mu_2,\ldots,\mu_p\}$. Then the distance spectrum of $G\{K_n\}$ consists of the numbers -1 of multiplicity $(n-2)p$, the roots of the equation*

$$\prod_{i=2}^{p}\left[x^3-(n(\mu_i-3)+\mu_i)x^2-2n(2\mu_i-1)x-2n\mu_i\right]=0$$

together with the three roots of

$$x^3-(n(k-3)+p(4n-2)+k)x^2-\left(p^2(5n-4)+2n(p+2k-1)\right)x-p^2(3n-2)-2nk=0. \quad (1)$$

Proof. By the definition of the cluster of two graphs, the distance matrix F of $G\{K_n\}$ can be written in the form

$$F=\begin{bmatrix} D & D+J & J_{1\times n-1}\otimes(D+2J)\\ D+J & D+2(J-I) & J_{1\times n-1}\otimes(D+3J-2I)\\ J_{n-1\times 1}\otimes(D+2J) & J_{n-1\times 1}\otimes(D+3J-2I) & I_{n-1}\otimes(D+4(J-I))+(J-I)_{n-1}\otimes(D+4J-3I)\end{bmatrix}$$

where J is the all-one matrix and I, the identity matrix of appropriate orders.

Let Y_j, $j=2,3,\ldots,n-1$ be the eigenvectors of J_{n-1} corresponding to zero. Then Y_j is orthogonal to the all one vector. Let $e^m_{p\times 1}$ is a $p\times 1$ column vector with the mth entry equals 1 and all other entries equal to zero.

Claim: $\Psi^m_j=\begin{bmatrix}0_{p\times1}\\0_{p\times1}\\Y_j\otimes e^m_{p\times1}\end{bmatrix}$ is an eigenvector of F with eigenvalues -1 for each $j=2,3,\ldots,n-1$ and $m=1,2,\ldots,p$. For

$$\begin{aligned}F\cdot\Psi^m_j&=\begin{bmatrix} D & D+J & J_{1\times n-1}\otimes(D+2J)\\ D+J & D+2(J-I) & J_{1\times n-1}\otimes(D+3J-2I)\\ J_{n-1\times 1}\otimes(D+2J) & J_{n-1\times 1}\otimes(D+3J-2I) & I_{n-1}\otimes[D+4(J-I)]+(J-I)_{n-1}\otimes(D+4J-3I)\end{bmatrix}\\
&\quad\times\begin{bmatrix}0_{p\times1}\\0_{p\times1}\\Y_j\otimes e^m_{p\times1}\end{bmatrix}\\
&=\begin{bmatrix}\left(J_{1\times n-1}\otimes(D+2J)\right)\cdot Y_j\otimes e^m_{p\times1}\\ \left(J_{1\times n-1}\otimes(D+3J-2I)\right)\cdot Y_j\otimes e^m_{p\times1}\\ \left(I_{n-1}\otimes[D+4(J-I)]+(J-I)_{n-1}\otimes(D+4J-3I)\right)\cdot Y_j\otimes e^m_{p\times1}\end{bmatrix}\\
&=\begin{bmatrix}0\\0\\I_{n-1}Y_j\otimes(D+4(J-I)-((D+4J-3I)))e^m_{p\times1}\end{bmatrix}\\
&=\begin{bmatrix}0\\0\\-1\cdot Y_j\otimes e^m_{p\times1}\end{bmatrix}\\
&=-1\,\Psi^m_j.\end{aligned}$$

Thus -1 is an eigenvalue of F with multiplicity $p(n-2)$.

Now consider the eigenvalue $\mu_i\neq k$ of D_G with an eigenvector X_i, $i=2,3,\ldots,p$. Let μ_{i_r}, $r=1,2,3$ be the three roots of the equation

$$x^3-(n(\mu_i-3)+\mu_i)x^2-2n(2\mu_i-1)x-2n\mu_i=0. \quad (2)$$

Claim: For each $i=2,3,\ldots,p$, the roots of Eq. (2), μ_{i_r}, $r=1,2,3$ are eigenvalues of F.

To prove the claim we investigate the condition under which $\Phi^r_i=\begin{bmatrix}t_rX_i\\X_i\\J_{n-1\times1}\otimes s_rX_i\end{bmatrix}$ becomes an eigenvector corresponding to μ_{i_r}, $r=1,2,3$ for F.

Now using $F\cdot\Phi^r_i=\mu_{i_r}\cdot\Phi^r_i$ and $X_i\neq 0$, we get the following.

$$t_r\mu_i+\mu_i+(n-1)s_r\mu_i=\mu_{i_r}t_r \quad (3)$$

$$t_r\mu_i+(\mu_i-2)+(n-1)(\mu_i-2)s_r\mu_i=\mu_{i_r} \quad (4)$$

$$t_r \mu_i + (\mu_i - 2) + s_r (\mu_i - 4) + (n-2) [s_r (\mu_i - 3)] = \mu_{i_r} s_r. \tag{5}$$

Now solving Eqs. (3) and (4) for t_r and s_r, we get

$$t_r = \frac{\mu_i \mu_{i_r}}{\mu_{i_r} (\mu_i - 2) + 2\mu_i}; \qquad s_r = \frac{\mu_{i_r} \left(\mu_{i_r} + 2\right) - 2\mu_i \left(\mu_{i_r} + 1\right)}{(n-1) \left(2\mu_i + \mu_{i_r} (\mu_i - 2)\right)}.$$

Now substituting these values in Eq. (5) yields a cubic in μ_{i_r} which is equivalent to Eq. (2), proving our claim. Thus forming eigenvectors of this type we get $p(n-2) + 3(p-1) = np + p - 3$ eigenvectors and there remains 3. By the construction, all eigenvectors are orthogonal to the all one vectors and hence the remaining three are of the form $\Omega = \begin{bmatrix} \alpha J_{p\times 1} \\ \beta J_{p\times 1} \\ J_{n-1\times 1} \otimes \gamma J_{p\times 1} \end{bmatrix}$ for some $(\alpha, \beta, \gamma) \neq (0, 0, 0)$.

Let ν be an eigenvalue with an eigenvector Ω, then the equation $F \cdot \Omega = \nu \Omega$ gives the following.

$$k\alpha + (k+p)\beta + (n-1)(k+2p)\gamma = \nu\alpha \tag{6}$$

$$(k+p)\alpha + (k + 2(p-1))\beta + (n-1)(k+3p-2)\gamma = \nu\beta \tag{7}$$

$$(k+2p)\alpha + (k+3p-2)\beta + ((n-1)(k+4p-3) - 1)\gamma = \nu\gamma. \tag{8}$$

Now $\alpha \neq 0$. Otherwise

$$\beta \left(nk^2 + pk(2pn + n - p) + p^2 (n(4p-2) - 3p + 2)\right) = 0$$

implies $\beta = 0$ and $\gamma = 0$. Therefore without loss of generality we can assume that $\alpha = 1$ and solving Eqs. (6)–(8) yields the cubic in (1). Hence the theorem. □

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